# Formulae — Introduction to Stochastic Processes — 2nd. Semester 2020/21 — IST

**Discrete and continuous univariate distributions**

| $X$ (r.v.) | Values | $P(X=x)$ or $f_X(x)$ | $E(X)$ | $V(X)$ | $M_X(t)$ or $E(X^k)$ | $P_X(s)$ |
|---|---|---|---|---|---|---|
| Binomial$(n,p)$ | $\{0,\ldots,n\}$ | $\binom{n}{x}p^x(1-p)^{n-x}$ | $np$ | $np(1-p)$ | $[pe^t+(1-p)]^n$ | $(1-p+ps)^n$ |
| HyperG$(N,M,n)$ | $\{\max\{0,n-N+M\},$ $\ldots,\min\{n,M\}\}$ | $\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$ | $n\frac{M}{N}$ | $n\frac{M}{N}\frac{N-M}{N}\frac{N-n}{N-1}$ | not interesting | not interesting |
| Geometric$(p)$ | $\mathbb{N}$ | $(1-p)^{x-1}p$ | $\frac{1}{p}$ | $\frac{1-p}{p^2}$ | $\frac{pe^t}{1-(1-p)e^t}$ | $\frac{ps}{1-(1-p)s}$ |
| Geometric$^\star(p)$ | $\mathbb{N}_0$ | $(1-p)^x p$ | $\frac{1-p}{p}$ | $\frac{1-p}{p^2}$ | $\frac{p}{1-(1-p)e^t}$ | $\frac{p}{1-(1-p)s}$ |
| NegativeBin$(r,p)$ | $\{r,r+1,\ldots\}$ | $\binom{x-1}{r-1}p^r(1-p)^{x-r}$ | $\frac{r}{p}$ | $\frac{r(1-p)}{p^2}$ | $\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$ | $\left[\frac{ps}{1-(1-p)s}\right]^r$ |
| NegativeBin$^\star(r,p)$ | $\{0,1,\ldots\}$ | $\binom{y+r-1}{r-1}p^r(1-p)^y$ | $\frac{r(1-p)}{p}$ | $\frac{r(1-p)}{p^2}$ | $\left[\frac{p}{1-(1-p)e^t}\right]^r$ | $\left[\frac{p}{1-(1-p)s}\right]^r$ |
| Poisson$(\lambda)$ | $\mathbb{N}_0$ | $e^{-\lambda}\frac{\lambda^x}{x!}$ | $\lambda$ | $\lambda$ | $e^{\lambda(e^t-1)}$ | $e^{-\lambda(1-s)}$ |
| Uniform$(\{1,\ldots,n\})$ | $\{1,\ldots,n\}$ | $\frac{1}{n}$ | $\frac{n+1}{2}$ | $\frac{n^2-1}{12}$ | $\frac{e^t(1-e^{tn})}{n(1-e^t)}$ | $\frac{s(1-s^n)}{n(1-s)}$ |
| Beta$(\alpha,\beta)$ | $[0,1]$ | $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$ | $\frac{\alpha}{\alpha+\beta}$ | $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$ | $1+\sum_{k=1}^{+\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}$ | — |
| Cauchy$(\mu,\sigma)$ | $\mathbb{R}$ | $\frac{1}{\pi\sigma}\frac{1}{1+\left(\frac{x-\mu}{\sigma}\right)^2}$ | nonexistent | nonexistent | nonexistent | — |
| $\chi^2_{(n)}$ | $\mathbb{R}_0^+$ | $\frac{(\frac{1}{2})^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}x^{\frac{n}{2}-1}e^{-\frac{x}{2}}$ | $n$ | $2n$ | $\left(\frac{\frac{1}{2}}{\frac{1}{2}-t}\right)^{\frac{n}{2}}, \quad t<\frac{1}{2}$ | — |
| Exponential$(\lambda)$ | $\mathbb{R}_0^+$ | $\lambda e^{-\lambda x}$ | $\frac{1}{\lambda}$ | $\frac{1}{\lambda^2}$ | $\frac{\lambda}{\lambda-t}, \quad t<\lambda$ | — |
| Gamma$(\alpha,\lambda)$ | $\mathbb{R}_0^+$ | $\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}$ | $\frac{\alpha}{\lambda}$ | $\frac{\alpha}{\lambda^2}$ | $\left(\frac{\lambda}{\lambda-t}\right)^\alpha, \quad t<\lambda$ | — |
| LogNormal$(\mu,\sigma^2)$ | $\mathbb{R}^+$ | $\frac{1}{\sqrt{2\pi}\sigma x}e^{-\frac{(\log x-\mu)^2}{2\sigma^2}}$ | $e^{\mu+\frac{\sigma^2}{2}}$ | $\left(e^{\sigma^2}-1\right)e^{2\mu+\sigma^2}$ | $E(X^k)=e^{k\mu+\frac{k^2\sigma^2}{2}}$ | — |
| Normal$(\mu,\sigma^2)$ | $\mathbb{R}$ | $\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ | $\mu$ | $\sigma^2$ | $e^{\mu t+\frac{(t\sigma)^2}{2}}$ | — |
| Rayleigh$(\sigma)$ | $\mathbb{R}_0^+$ | $\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}$ | $\sigma\sqrt{\frac{\pi}{2}}$ | $\frac{4-\pi}{2}\sigma^2$ | $E(X^k)=\left(\sqrt{2}\sigma\right)^k\Gamma\left(1+\frac{k}{2}\right)$ | — |
| Uniform$(a,b)$ | $[a,b]$ | $\frac{1}{b-a}$ | $\frac{a+b}{2}$ | $\frac{(b-a)^2}{12}$ | $\frac{e^{bt}-e^{at}}{(b-a)t}, \; t\neq 0$ | — |
| Weibull$(\alpha,\beta)$ | $\mathbb{R}_0^+$ | $\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}e^{-\left(\frac{x}{\alpha}\right)^\beta}$ | $\alpha\Gamma\left(1+\frac{1}{\beta}\right)$ | $\alpha^2\left[\Gamma\left(1+\frac{2}{\beta}\right)-\Gamma\left(1+\frac{1}{\beta}\right)^2\right]$ | $E(X^k)=\alpha^k\Gamma\left(1+\frac{k}{\beta}\right)$ | — |

$\Gamma(\alpha)=\int_0^{+\infty}x^{\alpha-1}e^{-x}dx,\ \alpha>0;\quad \Gamma(n)=(n-1)!,\ n\in\mathbb{N};\quad \Gamma(\alpha+1)=\alpha\Gamma(\alpha),\ \alpha>0;\quad \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

**Relating c.d.f.**

$F_{NegativeBin(r,p)}(x) = 1 - F_{Binomial(x,p)}(r-1)$

$F_{Erlang(n,\lambda)}(x) = 1 - F_{Poisson(\lambda x)}(n-1)$

$F_{Gamma(\alpha,\beta)}(x) = F_{\chi^2_{(2\alpha)}}(2\beta x)$

$F_{Beta(\alpha,\beta)}(x) = 1 - F_{Binomial(\alpha+\beta-1,x)}(\alpha-1)$

**Moment/probability generating function; moments**

$M_X(t) = E\left(e^{tX}\right)$

$E(X^k) = \left.\frac{d^k M_X(t)}{dt^k}\right|_{t=0}$

$P_X(s) = E(s^X); \quad P(X=k) = \frac{1}{k!} \times \left.\frac{d^k P_X(s)}{ds^k}\right|_{s=0}$

$E[X(X-1)\cdots(X-k+1)] = \left.\frac{d^k P_X(s)}{ds^k}\right|_{s=1},\ k \in \mathbb{N}$

$E(X) = \int_0^{+\infty} [1 - F_X(x)]\, dx, \ \text{ for } X \geq 0$

$E(X^k) = \int_0^{+\infty} k x^{k-1} [1 - F_X(x)]\, dx, \ \text{ for } X \geq 0$

$SC(X) = \frac{E\{[X-E(X)]^3\}}{[SD(X)]^3}$

$KC(X) = \frac{E\{[X-E(X)]^4\}}{[SD(X)]^4} - 3$

**Multinomial distribution**

$P(N_1 = n_1, \ldots, N_d = n_d) = \frac{n!}{\prod_{i=1}^d n_i!} \times \prod_{i=1}^d p_i^{n_i}$

$\{(n_1, \ldots, n_d) \in I\!N_0^d : \sum_{i=1}^d n_i = n\}$

$M_{N_1,\ldots,N_{d-1}}(t_1, \ldots, t_{d-1}) = \left[\left(\sum_{i=1}^{d-1} p_i\, e^{t_i}\right) + p_d\right]^n$

$N_i \sim \text{Binomial}(n, p_i); \quad Cov(N_i, N_j) = -n\, p_i\, p_j,\ i \neq j$

$M_{\underline{X}}(\underline{t}) = E\left[\exp\left(\sum_{i=1}^n t_i\, X_i\right)\right]$

$E\left(\prod_{i=1}^n X_i^{k_i}\right) = \left.\frac{\partial^{\sum_{i=1}^n k_i} M_{\underline{X}}(\underline{t})}{\partial t_1^{k_1} \ldots \partial t_n^{k_n}}\right|_{\underline{t}=\underline{0}}$

**Functions of r.v.**

$F_{Y=g(X)}(y) = P[X \in g^{-1}((-\infty, y])]$

$f_{Y=g(X)}(y) = f_X[g^{-1}(y)] \times \left|\frac{dg^{-1}(y)}{dy}\right|$

**Hierarchical models resulting from mixtures**

$P(X=x) = \sum_y P(X=x|Y=y) \times P(Y=y)$

$P(X=x) = \int_{I\!R_Y} P(X=x|Y=y) \times f_Y(y)\, dy$

$E[g(X)] = E\{E[g(X)|Y)]\}$

$V[g(X)] = V\{E[g(X)|Y]\} + E\{V[g(X)|Y]\}$

**Functions of random vectors**

$F_{\underline{Y}=\underline{g}(\underline{X})}(\underline{y}) = P\left[\underline{X} \in \underline{g}^{-1}\left(\prod_{i=1}^m (-\infty, y_i]\right)\right]$

$f_{\underline{Y}=\underline{g}(\underline{X})}(\underline{y}) = f_{\underline{X}}\left[\underline{g}^{-1}(\underline{y})\right] \times |J(\underline{y})|$

$$J(\underline{y}) = \begin{vmatrix} \frac{\partial g_1^{-1}(\underline{y})}{\partial y_1} & \ldots & \frac{\partial g_1^{-1}(\underline{y})}{\partial y_n} \\ \vdots & \ldots & \vdots \\ \frac{\partial g_n^{-1}(\underline{y})}{\partial y_1} & \ldots & \frac{\partial g_n^{-1}(\underline{y})}{\partial y_n} \end{vmatrix}$$

$f_{X+Y}(z) = \int_{-\infty}^{+\infty} f_{X,Y}(z-y, y)\, dy$

$f_{X-Y}(u) = \int_{-\infty}^{+\infty} f_{X,Y}(u+y, y)\, dy$

$f_{XY}(v) = \int_{-\infty}^{+\infty} f_{X,Y}(v/y, y) \times \frac{1}{|y|}\, dy$

$f_{X/Y}(w) = \int_{-\infty}^{+\infty} f_{X,Y}(wy, y) \times |y|\, dy$

**Order statistics**

$P[X_{(n-k+1)} > x] = 1 - F_{Binomial(n, 1-F_X(x))}(k-1)$

$f_{X_{(1)},\ldots,X_{(n)}}(x_{(1)}, \ldots, x_{(n)}) = n! \times \prod_{i=1}^n f_X(x_{(i)})$

$F_{X_{(i)}}(x) = 1 - F_{Binomial(n, F_X(x))}(i-1)$

$f_{X_{(i)}}(x) = \frac{n!}{(i-1)!(n-i)!} [F_X(x)]^{i-1} [1 - F_X(x)]^{n-i} f_X(x)$

$f_{\left(X_{(i)}, X_{(j)}\right)}(x, y) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} [F_X(x)]^{i-1} [F_X(y) - F_X(x)]^{j-i-1} [1 - F_X(y)]^{n-j} f_X(x) f_X(y),\ x < y$

**Cap. 0**

BERNOULLI PROCESS

$\{X_n : n \in \mathbb{N}\} \sim BP(p)$ $\quad S_n = \sum_{i=1}^n X_i \sim \text{Binomial}(n, p)$, $\quad T_k = \min\{n \in \mathbb{N} : S_n = k\} \sim \text{NegativeBin}(k, p)$

$U_k = T_k - T_{k-1} \overset{i.i.d.}{\sim} \text{Geometric}(p),\ k \in \mathbb{N}$

$S_m | S_n = k \sim \text{HyperG}(n, m, k),\ 0 \leq m \leq n,\ 0 \leq k \leq n$

**Cap. 1**

A FEW PROPERTIES OF THE EXPONENTIAL DISTRIBUTION

$X_i \overset{indep}{\sim} \text{Exponential}(\lambda_i),\ i=1,\dots,n\ \Rightarrow$ a) $\min_{i=1,\dots,n}\{X_i\} \sim \text{Exponential}(\sum_{i=1}^n \lambda_i)$

b) $P\left(X_j = \min_{i=1,\dots,n}\{X_i\}\right) = \frac{\lambda_j}{\sum_{i=1}^n \lambda_i}$ c1) $\sum_{i=1}^n X_i \sim \text{Hypo-exp.}(\lambda_1,\dots,\lambda_n),\ \lambda_i \neq \lambda_j\ (i\neq j)$

c2) $f_{\sum_{i=1}^n X_i}(x) = \sum_{i=1}^n C_{i,n} \times \lambda_i\, e^{-\lambda_i x},\ C_{i,n} = \prod_{j\neq i} \frac{\lambda_j}{\lambda_j - \lambda_i}$

d1) $X \sim \text{Hyper-exp.}(\lambda_1,\dots,\lambda_n; p_1,\dots,p_n),\ \lambda_i \neq \lambda_j\ (i\neq j)$ d2) $f_X(x) = \sum_{i=1}^n p_i\, f_{X_i}(x)$

HOMOGENOUS PP (independent and stationary increments)

$\{N(t): t\geq 0\} \sim PP(\lambda)$ $S_n = \min\{t\geq 0 : N(t)=n\}$; $N(t)\geq n \Leftrightarrow S_n \leq t$; $N(t) \sim \text{Poisson}(\lambda t)$

$S_n \sim \text{Erlang}(n,\lambda)$; $F_{S_n}(t) = 1 - F_{Poisson(\lambda t)}(n-1)$; $(N(s)\mid N(t)=n) \sim \text{Binomial}(n, s/t),\ 0<s<t$

$P[N(t_1)=k_1,\dots,N(t_n)=k_n] = \prod_{j=1}^n \frac{e^{-\lambda(t_j-t_{j-1})}\,[\lambda(t_j-t_{j-1})]^{k_j-k_{j-1}}}{(k_j-k_{j-1})!},\ 0=t_0<t_1<\dots<t_n,\ 0=k_0\leq k_1\leq\dots\leq k_n$

$\{N_i(t): t\geq 0\} \overset{indep}{\sim} PP(\lambda_i),\ i=1,2 \quad\Rightarrow\quad P\left[S_n^{(1)} < S_m^{(2)}\right] = 1 - F_{Binomial(n+m-1,\frac{\lambda_1}{\lambda_1+\lambda_2})}(n-1)$

$\{X_n = S_n - S_{n-1} : n\in\mathbb{N}\} \overset{i.i.d.}{\sim} \text{Exp}(\lambda)$ $(S_1 \mid N(t)=1) \sim \text{Uniform}(0,t)$

$Y_i \overset{i.i.d.}{\sim} \text{Uniform}(0,t),\ i=1,\dots,n$ $(S_1,\dots,S_n \mid N(t)=n) \sim (Y_{(1)},\dots,Y_{(n)})$

$N_1(t) = \#$registered events in $(0,t]$, under the non-homogeneous Bernoulli splitting mechanism associated with a $p: \mathbb{R}_0^+ \to [0,1]$

$N_1(t) \sim \text{Poisson}\left(\lambda \int_0^t p(s)\,ds\right)$

NON-HOMOGENOUS PP (independent increments)

$\{N(t): t\geq 0\} \sim NHPP(\lambda(t))$ $m(t) = \int_0^t \lambda(z)\,dz$ $N(t+s)-N(s) \sim \text{Poisson}\left(m(t+s)-m(s)\right)$

$P\left[\bigcap_{j=1}^n N(t_j)=k_j\right] = \prod_{j=1}^n \frac{e^{-[m(t_j)-m(t_{j-1})]}\,[m(t_j)-m(t_{j-1})]^{k_j-k_{j-1}}}{(k_j-k_{j-1})!},\ 0=t_0<t_1<\dots<t_n,\ 0=k_0\leq k_1\leq\dots\leq k_n$

$(N(s)\mid N(t)=n) \sim \text{Binomial}(n, \frac{m(s)}{m(t)}),\ 0<s<t,\ n\in\mathbb{N}$

$F_{S_n}(t) = 1 - F_{Poisson(m(t))}(n-1),\ n\in\mathbb{N}$ $P(X_{n+1} = S_{n+1}-S_n > t) = \int_0^{+\infty} \lambda(s) e^{-m(t+s)} \frac{[m(s)]^{n-1}}{(n-1)!}\,ds,\ n\in\mathbb{N}$

$Y_i \overset{i.i.d.}{\sim} Y$, where $P(Y\leq u) = \frac{m(u)}{m(t)}$, for $0\leq u\leq t$ $(S_1,\dots,S_n \mid N(t)=n) \sim (Y_{(1)},\dots,Y_{(n)})$

CONDITIONAL PP (stationary increments)

$\{N(t): t\geq 0\} \sim ConditionalPP(G)$ $P[N(t+s)-N(s)=n] = \int_0^{+\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!}\,dG(\lambda)$

COMPOUND PP (independent and stationary increments)

$\{X(t) = \sum_{i=1}^{N(t)} Y_i : t\geq 0\} \sim CompoundPP(\lambda, Y)$ $E[X(t)] = \lambda t \times E(Y)$; $V[X(t)] = \lambda t \times E(Y^2)$

**Cap. 2**

RENEWAL PROCESSES

$\{N(t): t\geq 0\} \sim RP$ with inter-renewal distribution $F$ and mean $\mu$; $F_n(t) = P(S_n \leq t)$

$P[N(t)=n] = F_n(t) - F_{n+1}(t)$ $m(t) = E[N(t)] = \sum_{n=1}^{+\infty} F_n(t)$

$m(t) = F(t) + \int_0^t m(t-x)\,dF(x)$ $\tilde{m}(s) = \int_{0^-}^{+\infty} e^{-st}\,dm(t) = \frac{\tilde{F}(s)}{1-\tilde{F}(s)}$

$H(t) = D(t) + \int_0^t H(t-x)\,dF(x)$ $|D(t)| < +\infty\ \Rightarrow\ H(t) = D(t) + \int_0^t D(t-x)\,dm(x)$

$H(t) = D(t) + \int_0^t D(t-x)\,dm(x)$ $\tilde{D}(s) = \int_{0^-}^{+\infty} e^{-st}\,dD(t)$ exists; $\tilde{H}(s) = \frac{\tilde{D}(s)}{1-\tilde{F}(s)}$

## Important Laplace transforms

| $f(t)$ | $f^\star(s)=\int_0^\infty e^{-st}f(t)\,dt$ | $g(t)$ | $g^\star(s)=\int_0^\infty e^{-st}g(t)\,dt$ |
|---|---|---|---|
| $1$ | $1/s$ | $a\,f(t)+b\,h(t)$ | $a\,f^\star(s)+b\,h^\star(s)$ |
| $t^n$ | $n!/s^{n+1}$ | $\frac{df(t)}{dt}$ | $s\,f^\star(s)-f(0)$ |
| $\frac{t^{n-1}e^{-at}}{(n-1)!}$ | $1/(s+a)^n$ | $e^{-at}f(t)$ | $f^\star(s+a)$ |
| $\sin(at)$ | $a/(s^2+a^2)$ | $\int_0^t f(u)du$ | $f^\star(s)/s$ |
| $\frac{e^{-at}-e^{-bt}}{b-a}$ | $1/[(s+a)(s+b)]$ | $t^n f(t)$ | $(-1)^n\frac{d^n}{ds^n}f^\star(s)$ |

## Limit theorems et al.

$P\left[N(t)<n\right]\simeq\Phi\left(\frac{n-t/\mu}{\sqrt{t\sigma^2/\mu^3}}\right)$

SLLN for renewal processes: $\frac{N(t)}{t}\overset{w.p.1}{\to}\frac{1}{\mu}$

Elementary renewal theorem: $\lim_{t\to+\infty}\frac{m(t)}{t}=\frac{1}{\mu}$

Key renewal th.: $D(t)$ dRi, $F$ not lattice, $H(t)=D(t)+\int_0^t D(t-x)\,dm(x)\ \Rightarrow\ \lim_{t\to+\infty}H(t)=\frac{1}{\mu}\int_0^{+\infty}D(y)\,dy$

Blackwell's th.: $F$ not lattice $\Rightarrow$ $\lim_{t\to+\infty}[m(t+a)-m(t)]=\frac{a}{\mu}$

$F$ lattice with period $d$ $\Rightarrow$ $\lim_{n\to+\infty}E$[number of renewals **at** $nd$] $=\frac{d}{\mu}$

## Recurrence times

$A(t)=t-S_{N(t)},\quad Y(t)=S_{N(t)+1}-t,\quad X_{N(t)+1}=A(t)+Y(t)$

$E\left[S_{N(t)+1}\right]=\mu\times[m(t)+1]$

$X_{N(t)+1}\geq_{st}X_i$, $i\in\mathbb{N}$ (inspection paradox)

$\frac{A(t)}{t}\overset{w.p.1}{\to}0,\quad \lim_{t\to 0}\frac{E[Y(t)]}{t}=0$

$\lim_{t\to+\infty}E[Y(t)]=\frac{E(X^2)}{2\mu}$

$\lim_{t\to+\infty}E[A(t)]=\frac{E(X^2)}{2\mu}$

$\lim_{t\to+\infty}E[X_{N(t)+1}]=\frac{E(X^2)}{\mu}\geq E(X)$

$\lim_{t\to+\infty}P[Y(t)\leq x]=\lim_{t\to+\infty}P[A(t)\leq x]=F_e(x)=\frac{\int_0^x[1-F(u)]\,du}{\mu}$ (equilibrium distribution)

## Reward renewal processes

$\{(X_n,R_n):n\in\mathbb{N}\}\overset{i.i.d.}{\sim}(X,R),\quad R(t)=\sum_{n=1}^{N(t)}R_n$

$E(X),E(R)<+\infty\ \Rightarrow$
a) $\frac{R(t)}{t}\overset{w.p.1}{\to}\frac{E(R)}{E(X)}$
b) $\lim_{t\to+\infty}\frac{E[R(t)]}{t}=\frac{E(R)}{E(X)}$

## Alternating renewal processes

$$\{(U_n,D_n):n\in\mathbb{N}\}\overset{i.i.d.}{\sim}(U,D),\quad Z(t)=\begin{cases}1, & \text{if } \exists n\in\mathbb{N}: S_n=\sum_{i=1}^n X_i=\sum_{i=1}^n(U_i+D_i)\leq t<S_n+U_{n+1}\\ 0, & \text{if } \exists n\in\mathbb{N}: S_n+U_{n+1}\leq t<S_{n+1}\end{cases}$$

$E(U_n+D_n)<+\infty$, $F$ not lattice $\Rightarrow$ $\lim_{t\to+\infty}P[Z(t)=1]=\frac{E(U)}{E(U)+E(D)}$

## Delayed renewal processes

$\{N_D(t):t\geq 0\}$, with $X_1\sim G$, $X_i\sim F$, $i=2,3,\dots$, and $(G\star F_{n-1})(t)=P(S_n\leq t)=\int_0^t G(t-x)\,dF_{n-1}(x)$

$P[N_D(t)=n]=P(S_n\leq t)-P(S_{n+1}\leq t)=(G\star F_{n-1})(t)-(G\star F_n)(t)$

$m_D(t)=\sum_{n=1}^{+\infty}(G\star F_{n-1})(t)$

$\tilde{m}_D(s)=\frac{\tilde{G}(s)}{1-\tilde{F}(s)}$

## Regenerative processes

$\{X(t):t\geq 0\}$, with state space $\mathbb{N}_0$ and $S_1\sim F$

$U_j=$ time spent in state $j$ during $[0,S_1)$

$F$ not lattice, $E(S_1)<+\infty\ \Rightarrow\ \lim_{t\to+\infty}P[X(t)=j]=\frac{E(U_j)}{E(S_1)}=P_j$

$E(S_1)<+\infty\ \Rightarrow\ \lim_{t\to+\infty}\frac{\text{amount of time state } j \text{ during } (0,t)}{t}\overset{w.p.1}{=}P_j$

**Cap. 3**

Definitions and examples

$\{X_n : n \in \mathbb{N}_0\},\ P(X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1},\dots,X_0=i_0) = P(X_{n+1}=j \mid X_n=i) = P(X_1=j \mid X_0=i) = P_{ij}$

$\mathbf{P} = [P_{ij}]_{i,j\in\mathcal{S}},\quad \underline{\alpha} = [\alpha_i]_{i\in\mathcal{S}} = [P(X_0=i)]_{i\in\mathbb{S}}$

Chapman-Kolmogorov equations; marginal and joint distributions

$P_{ij}^{n+m} = P(X_{n+m}=j \mid X_0=i) = \sum_{k\in\mathcal{S}} P_{ik}^n P_{kj}^m \qquad \mathbf{P}^n = \left[P_{ij}^n\right]_{i,j\in\mathcal{S}},\quad \underline{\alpha}^n = [P(X_n=j)]_{j\in\mathcal{S}} = \underline{\alpha}\,\mathbf{P}^n$

$P(X_{n_1}=i_{n_1},\dots,X_{n_k}=i_{n_k}) = \left(\sum_{i\in\mathcal{S}} \alpha_i \times P_{i,i_{n_1}}^{n_1}\right) \times \prod_{j=2}^k P_{i_{n_{j-1}},i_{n_j}}^{n_j-n_{j-1}},\quad 0\le n_1<n_2<\cdots<n_k,\ i_{n_1},\dots,i_{n_k}\in\mathcal{S}$

Classification of states; recurrent and transient states

$f_{ij}^n = P(X_n=j, X_{n-1}\neq j,\dots,X_1\neq j \mid X_0=i) \qquad T_i = \min\{n\in\mathbb{N} : X_n=i \mid X_0=i\}$

$f_{ij} = \sum_{n=1}^{+\infty} f_{ij}^n,\quad f_i \equiv f_{ii} = \sum_{n=1}^{+\infty} f_{ii}^n = P(T_i<+\infty)$ recurrent if $f_i=1$, transient if $f_i<1$

$\mu_{ii} = E(T_i) = \sum_{n=1}^{+\infty} n\times f_{ii}^n$ ($i$ recurrent) positive rec. if $\mu_{ii}<+\infty$, null rec. if $\mu_{ii}=+\infty$

Limit behavior of irreducible (aperiodic) Markov chains

$P_j = \sum_{i\in\mathcal{S}} P_i P_{ij},\ j\in\mathcal{S}$ (stationary dist.); $\lim_{n\to+\infty} P_{ij}^n = \pi_j > 0,\ i,j\in\mathcal{S}$ (if all states are posit. rec. aper.)

$\pi_j = \sum_{i\in\mathcal{S}} \pi_i P_{ij},\ j\in\mathcal{S},\quad \sum_{j\in\mathcal{S}} \pi_j = 1;\qquad \mu_{jj} = \frac{1}{\pi_j};\qquad \underline{\pi} = \underline{\pi}\mathbf{P},\ \underline{\pi}\,\underline{1} = 1;\quad \underline{\pi} = [\pi_j]_{j\in\mathcal{S}} = \underline{1}\times(\mathbf{I}-\mathbf{P}+\mathbf{ONE})^{-1}$

Markov chains with costs/rewards

$\lim_{N\to+\infty} \frac{1}{N+1} E\left[\sum_{n=0}^N c(X_n) \mid X_0=i\right] = \sum_{j\in\mathcal{S}} \pi_j c(j) = \underline{\pi}\times\underline{c}$ (long-run expected cost per time unit)

$\phi(i) = E\left[\sum_{n=0}^{+\infty} \alpha^n c(X_n) \mid X_0=i\right]$ (expected total discounted cost incurred over..., starting at state $i$)

$\phi(i) = c(i) + \alpha\sum_{j\in\mathcal{S}} P_{ij}\phi(j),\ i\in\mathcal{S} \qquad \underline{\phi} = [\phi(i)]_{i\in\mathcal{S}} = (\mathbf{I}-\alpha\mathbf{P})^{-1}\times\underline{c}$

Time reversible Markov chains

$\{X_{n-m} : m\in\mathbb{Z}\},\quad Q_{ij} = \frac{\pi_j\times P_{ji}}{\pi_i} \qquad \pi_i\times P_{ij} = \pi_j\times P_{ji},\quad i,j\in\mathcal{S}$

$P_{i,i_1}\times P_{i_1,i_2}\times\cdots\times P_{i_k,i} = P_{i,i_k}\times\cdots\times P_{i_2,i_1}\times P_{i_1,i}$, for any $i, i_1, i_2, \dots, i_k$, $k\in\mathbb{N}$ (Kolmogorov's criterion)

Branching processes

$\{X_n : n\in\mathbb{N}_0\},\quad X_0=1,\quad X_n = \sum_{l=1}^{X_{n-1}} Z_l,\ n\in\mathbb{N},\quad \{Z_l : l\in\mathbb{N}\}$ i.i.d., non-negat. r.v., $P_j = P(Z_l=j),\ j\in\mathbb{N}_0$

$P_n(s) = E(s^{X_n}),\quad P(s) = E(s^{Z_l}),\quad s\in[0,1] \qquad P_{n+1}(s) = P_n[P(s)] = P[P_n(s)],\ n\in\mathbb{N},\ s\in[0,1]$

$\mu = E(Z_l),\quad \sigma^2 = V(Z_l),\quad E(X_n \mid X_0=1) = \mu^n \qquad V(X_n \mid X_0=1) = \begin{cases} \sigma^2\mu^{n-1}\times\frac{\mu^n-1}{\mu-1}, & \text{if } \mu\neq 1 \\ n\sigma^2, & \text{if } \mu=1, \end{cases}$

$\pi = \lim_{n\to+\infty} P(X_n=0 \mid X_0=1)$ if $\mu\le 1$, $\pi=1$; if $\mu>1$, $\pi = \sum_{j=0}^{+\infty} \pi^j\times P_j$

First passage times; absorption probabilities

$f_{ij}^n = P(X_n=j, X_{n-1}\neq j,\dots,X_1\neq j \mid X_0=i) \qquad \underline{f}_j^n = [f_{ij}^n]_{i\in\mathcal{S}}$

$f_{ij}^n = \begin{cases} P_{ij}, & n=1 \\ \sum_{k\neq j} P_{ik} f_{kj}^{n-1}, & n=2,3,\dots \end{cases} \qquad \underline{f}_j^n = \begin{cases} \underline{f}_j^1 = [P_{ij}]_{i\in\mathcal{S}}, & n=1 \\ {}^{(j)}\mathbf{P}\times\underline{f}_j^{n-1} = \left[{}^{(j)}\mathbf{P}\right]^{n-1}\times\underline{f}_j^1, & n=2,3,\dots \end{cases}$

$\mathcal{S} = T \cup C_1 \cup C_2 \cup \cdots, \quad \mathbf{Q} = [Q_{ij}]_{i,j\in T}, \quad \mathbf{R} = [P_{kl}]_{k\in T,\, l\in \overline{T}}, \quad \tau = \inf\{n \in \mathbb{N}_0 : X_n \notin T\}, \quad u_{ik} = P(X_\tau = k \mid X_0 = i)$

$\mathbf{U} = [u_{ik}]_{i\in T,\, k\in\overline{T}} = (\mathbf{I} - \mathbf{Q})^{-1} \times \mathbf{R}$ $\qquad$ $[E(\tau \mid X_0 = i)]_{i\in T} = (\mathbf{I} - \mathbf{Q})^{-1} \times \underline{1}$

$\left[E[\sum_{n=0}^{\tau-1} g(X_n) \mid X_0 = i]\right]_{i\in T} = (\mathbf{I} - \mathbf{Q})^{-1} \times \underline{g}$

**Cap. 4**

Definitions and examples

$\{X(t) : t \geq 0\}, \quad P[X(t+s) = j \mid X(s) = i,\, X(u) = x(u),\, 0 \leq u < s] = P[X(t+s) = j \mid X(s) = i]$

$P[X(t+s) = j \mid X(s) = i] = P[X(t) = j \mid X(0) = i] = P_{ij}(t), \quad \mathbf{P}(t) = [P_{ij}(t)]_{i,j\in\mathcal{S}}, \quad \underline{\alpha} = [\alpha_i]_{i\in\mathcal{S}} = [P[X(0) = i]]_{i\in\mathcal{S}}$

Properties of the transition matrix; Chapman-Kolmogorov equations

$P_{ij}(t+s) = \sum_{k\in\mathcal{S}} P_{ik}(t) \times P_{kj}(s) \quad \mathbf{P}(t+s) = \mathbf{P}(t) \times \mathbf{P}(s) = \mathbf{P}(s) \times \mathbf{P}(t) \quad P[X(t) = j] = \sum_{i\in\mathcal{S}} \alpha_i \times P_{ij}(t),\ j \in \mathcal{S}$

$[P[X(t) = j)]_{j\in\mathcal{S}} = \underline{\alpha}\, \mathbf{P}(t) \quad P\left[X(t_1) = x(t_1), \ldots, X(t_k) = x(t_k)\right] = \left[\sum_{i\in\mathcal{S}} \alpha_i\, P_{i,x(t_1)}(t_1)\right] \prod_{j=2}^{k} P_{x(t_{j-1}),x(t_j)}(t_j - t_{j-1})$

$q_{ij} = \nu_i \times P_{ij},\ i \neq j$ $\qquad$ $\lim_{h\to 0^+} \frac{P_{ij}(h)}{h} = q_{ij},\ i \neq j; \quad \lim_{h\to 0^+} \frac{1-P_{ii}(h)}{h} = \nu_i$

$\frac{d\,P_{ij}(t)}{dt} = \sum_{k\neq i} q_{ik} P_{kj}(t) - \nu_i P_{ij}(t)$ (backward eq.) $\qquad$ $\frac{d\,P_{ij}(t)}{dt} = \sum_{k\neq j} P_{ik}(t) q_{kj} - P_{ij}(t)\nu_j$ (forward eq.)

$r_{ij} = \begin{cases} q_{ij}, & i \neq j \\ -\nu_i, & i = j, \end{cases} \qquad \mathbf{R} = [r_{ij}]_{i,j\in\mathcal{S}}$ $\qquad$ $\frac{d\,\mathbf{P}(t)}{dt} = \left[\frac{d\,P_{ij}(t)}{dt}\right]_{i,j\in\mathcal{S}} = \mathbf{R} \times \mathbf{P}(t) = \mathbf{P}(t) \times \mathbf{R}$

Computing the transition matrix: finite state space

$\mathbf{P}(t) = e^{\mathbf{R}\,t} = \sum_{n=0}^{+\infty} \frac{\mathbf{R}^n\, t^n}{n!} = \lim_{n\to+\infty} \left(\mathbf{I} + \frac{\mathbf{R}\,t}{n}\right)^n$

Computing the transition matrix: infinite state space

$P_{ij}^\star(s) = \int_0^{+\infty} e^{-st} P_{ij}(t) dt, \quad i, j \in \mathcal{S}$

$\int_0^{+\infty} e^{-st} \frac{d\,P_{ij}(t)}{dt} dt = s \times P_{ij}^\star(s) - P_{ij}(0) = \sum_{k\neq j} P_{ik}^\star(s) \times q_{kj} - P_{ij}^\star(s) \times \nu_j$

Birth and death processes

$\frac{d\,P_{0j}(t)}{dt} = \lambda_0\, P_{1j}(t) - \lambda_0\, P_{0j}(t),\ j \in \mathbb{N}_0; \ \ \frac{d\,P_{ij}(t)}{dt} = \lambda_i\, P_{i+1,j}(t) + \mu_i\, P_{i-1,j}(t) - (\lambda_i + \mu_i)\, P_{ij}(t),\ i \in \mathbb{N},\ j \in \mathbb{N}_0$ (b. eq.)

$\frac{d\,P_{i0}(t)}{dt} = P_{i1}(t)\,\mu_1 - P_{i0}(t)\,\lambda_0,\ i \in \mathbb{N}_0; \ \ \frac{d\,P_{ij}(t)}{dt} = P_{i,j-1}(t)\,\lambda_{j-1} + P_{i,j+1}(t)\,\mu_{j+1} - P_{ij}(t)\,(\lambda_j + \mu_j),\ i \in \mathbb{N}_0,\ j \in \mathbb{N}$ (f. eq.)

$P_j(t) \equiv P[X(t) = j \mid X(0) = i]$ $\qquad$ $P(z,t) = E\left[z^{X(t)} \mid X(0) = i\right],\ |z| \leq 1$

$\sum_{j\in\mathcal{S}} z^j \times \frac{d\,P_j(t)}{dt} = \frac{\partial P(z,t)}{\partial t}$ $\qquad$ $\frac{\partial P(z,t)}{\partial z} = \sum_{j\in\mathcal{S}} j z^{j-1} \times P_j(t) = \sum_{j\in\mathcal{S}} (j+1) z^j \times P_{j+1}(t)$

$\sum_{j\in\mathcal{S}} z^j \times \frac{d\,P_j(t)}{dt} = \sum_{j\in\mathcal{S}} z^j \times [P_{j-1}(t)\,\lambda_{j-1} + P_{j+1}(t)\,\mu_{j+1} - P_j(t)\,(\lambda_j + \mu_j)]$ (forward eq.)

Limit behavior of CTMC

$\{X_n : n \in \mathbb{N}_0\}$ (embedded DTMC), $\quad \underline{\pi} = [\pi_j]_{j\in\mathcal{S}}$ (stationary distribution of the embedded DTMC)

$P_j = \lim_{t\to+\infty} P_{ij}(t)$ $\qquad$ $P_j = \frac{\frac{\pi_j}{\nu_j}}{\sum_{k\in\mathcal{S}} \frac{\pi_k}{\nu_k}}, \quad j \in \mathcal{S}$

$\underline{P} = [P_j]_{j\in\mathcal{S}}, \quad \underline{P} \times \mathbf{R} = \underline{0}, \quad \sum_{j\in\mathcal{S}} P_j = 1$ $\qquad$ $P_j \times \nu_j = \sum_{i\in\mathcal{S}} P_i \times q_{ij}, \quad j \in \mathcal{S}$

$P_0 = \left[1 + \sum_{n=1}^{+\infty} \frac{\lambda_0\,\lambda_1 \ldots \lambda_{n-1}}{\mu_1\,\mu_2 \ldots \mu_n}\right]^{-1}, \quad P_j = \frac{\lambda_{j-1}}{\mu_j} P_{j-1} = P_0 \times \frac{\lambda_0\,\lambda_1 \ldots \lambda_{j-1}}{\mu_1\,\mu_2 \ldots \mu_j},\ j \in \mathbb{N}$ (birth and death proc.)

**Queues**

| | |
|---|---|
| $L_s$ | $E(L_s) = \lambda_e E(W_s) = \lambda \times (1 - P_b) \times E(W_s)$ |
| | $E(L_s) = \frac{\lambda_e}{\mu} + E(L_q)$ |
| $L_q$ | $E(L_q) = \lambda_e E(W_q)$ |
| $W_s$ | $E(W_s) = \frac{1}{\mu} + E(W_q)$ |

**M/M/1**

| | |
|---|---|
| Rates | $\lambda_k = \lambda, \quad k \in \mathbb{N}_0$<br>$\mu_k = \mu, \quad k \in \mathbb{N} \qquad \left(\rho = \frac{\lambda}{\mu} < 1\right)$ |
| $L_s$ | $P(L_s = k) = \rho^k (1 - \rho), \quad k \in \mathbb{N}_0$<br>$E(L_s) = \frac{\rho}{(1-\rho)}$ |
| $L_q$ | $P(L_q = k) = \begin{cases} 1 - \rho^2, & k = 0 \\ \rho^{k+1}(1-\rho), & k \in \mathbb{N} \end{cases}$<br>$E(L_q) = \frac{\rho^2}{(1-\rho)}$ |
| $W_s$ | $(W_s \mid L_s = k) \sim \text{Gamma}(k+1, \mu),\ k \in \mathbb{N}_0$<br>$W_s \sim \text{Exponential}(\mu(1-\rho))$<br>$E(W_s) = \frac{1}{\mu(1-\rho)}$ |
| $W_q$ | $(W_q \mid L_s = k) \sim \text{Gamma}(k, \mu),\ k \in \mathbb{N}$<br>$F_{W_q}(t) = \begin{cases} 0, & t < 0 \\ 1 - \rho, & t = 0 \\ (1-\rho) + \rho \times F_{Exp(\mu(1-\rho))}(t), & t > 0 \end{cases}$<br>$(W_q \mid W_q > 0) \sim \text{Exponential}(\mu(1-\rho))$<br>$E(W_q) = \frac{\rho}{\mu(1-\rho)}$ |

**M/M/∞**

| | |
|---|---|
| Rates | $\lambda_k = \lambda, \quad k \in \mathbb{N}_0$<br>$\mu_k = k\mu, \quad k \in \mathbb{N} \qquad \left(\rho = \frac{\lambda}{\mu} < +\infty\right)$ |
| $L_s$ | $L_s \sim \text{Poisson}(\lambda/\mu)$ |

$L_q \stackrel{st}{=} 0, \quad W_s \sim \text{Exp}(\mu), \quad W_q \stackrel{st}{=} 0$

$X(t)$ = number of customers in the system at time $t$

$(X(t) \mid X(0) = 0) \sim \text{Poisson}(\lambda (1 - e^{-\mu t})/\mu)$

**M/G/∞**

$(X(t) \mid X(0) = 0) \sim \text{Poisson}\left(\lambda \int_0^t [1 - G(t-s)]\, ds\right)$

$\lim_{t \to +\infty} (X(t) \mid X(0) = 0) \sim \text{Poisson}(\lambda/\mu)$

**M/M/m**

| | |
|---|---|
| Rates | $\lambda_k = \lambda, \quad k \in \mathbb{N}_0$ <br> $\mu_k = \begin{cases} k\mu, & k = 1, \dots, m \\ m\mu, & k = m+1, m+2, \dots \end{cases} \qquad \left(\rho = \frac{\lambda}{m\mu} < 1\right)$ |
| $L_s$ | $P(L_s = k) = \begin{cases} \frac{m!}{k!}\,(1-\rho)(m\rho)^{k-m} C(m, m\rho), & k = 0, 1, \dots, m-1 \\ (1-\rho)\,\rho^{k-m}\, C(m, m\rho), & k = m, m+1, \dots \end{cases}$ <br> $C(m, m\rho) = P\left(L_s \geq m\right) = \frac{\frac{(m\rho)^m}{m!(1-\rho)}}{\sum_{j=0}^{m-1} \frac{(m\rho)^j}{j!} + \frac{(m\rho)^m}{m!(1-\rho)}}$ <br> $C(1, \rho) = \rho$ <br> $C(2, 2\rho) = \frac{2\rho^2}{1+\rho}$ <br> $E(L_s) = m\rho + \frac{\rho}{1-\rho} C(m, m\rho)$ |
| $L_q$ | $P(L_q = k) = \begin{cases} 1 - \rho\, C(m, m\rho), & k = 0 \\ (1-\rho)\,\rho^k\, C(m, m\rho), & k \in \mathbb{N} \end{cases}$ <br> $E(L_q) = \frac{\rho}{1-\rho} C(m, m\rho)$ |
| $W_s$ | $(W_s \mid L_s = k) \sim \begin{cases} \text{Exp}(\mu), \; k = 0, \dots, m-1, \\ \text{Exp}(\mu) \star \text{Gamma}(k - m + 1, m\mu), \; k = m, m+1, \dots, \end{cases}$ <br> $1 - F_{W_s}(t) = \begin{cases} \left[1 + \mu t\, C(m, m\rho)\right] e^{-\mu t}, & t \geq 0, \quad \rho = \frac{m-1}{m} \\ \left[1 + \frac{e^{\mu[1-m(1-\rho)]t} - 1}{1 - m(1-\rho)} \times C(m, m\rho)\right] e^{-\mu t}, & t \geq 0, \quad \rho \neq \frac{m-1}{m} \end{cases}$ <br> $E(W_s) = \frac{1}{\mu} + \frac{C(m, m\rho)}{m\mu(1-\rho)}$ |
| $W_q$ | $(W_q \mid L_s = k) \sim \text{Gamma}(k - m + 1, m\mu), \; k = m, m+1, \dots$ <br> $(W_q \mid W_q > 0) \sim \text{Exponential}(m\mu(1-\rho))$ <br> $1 - F_{W_q}(t) = \begin{cases} 1, & t < 0 \\ C(m, m\rho), & t = 0 \\ C(m, m\rho) \times [1 - F_{Exp(m\mu(1-\rho))}(t)], & t > 0 \end{cases}$ <br> $E(W_q) = \frac{C(m, m\rho)}{m\mu(1-\rho)}$ |

**M/M/m/m**

| | |
|---|---|
| Rates | $\lambda_k = \begin{cases} \lambda, & k = 0, 1, \dots, m-1 \\ 0, & k = m, m+1, \dots \end{cases}$ <br> $\mu_k = \begin{cases} k\mu, & k = 1, \dots, m \\ 0, & k = m+1, m+2, \dots \end{cases} \qquad \left(\rho = \frac{\lambda}{m\mu} < +\infty\right)$ |
| $L_s$ | $P(L_s = k) = \begin{cases} \frac{\frac{(m\rho)^k}{k!}}{\sum_{j=0}^{m} \frac{(m\rho)^j}{j!}} = \frac{m!}{k!\,(m\rho)^{m-k}} \times B(m, m\rho), & k = 0, 1, \dots, m \\ 0, & k = m+1, m+2, \dots \end{cases}$ <br> $B(m, m\rho) = P(L_s = m) = \frac{\frac{(m\rho)^m}{m!}}{\sum_{j=0}^{m} \frac{(m\rho)^j}{j!}}$ <br> $E(L_s) = m\rho\,[1 - B(m, m\rho)]$ |
| $L_q \overset{st}{=} 0, \quad W_s \sim \text{Exp}(\mu), \quad W_q \overset{st}{=} 0$ | |